Extremal Functions of Forbidden Matrices

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Meghal Gupta
Mentor: Jesse Geneson

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$0, 1...k$-matrix

Definition

$0, 1...k$-matrix: A matrix where all the entries are in $\{0, 1...k\}$.

Example

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

is a $0, 1$-matrix. It is also a $0, 1...k$-matrix for all $k > 1$

Containment and Avoidance for 0,1-Matrices

A

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

B

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Definition

A *contains* B: We can delete rows and columns in A, and replace 1's with 0's to end up with B.

Containment and Avoidance for 0,1-Matrices

$$\underset{\text{A}}{\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}} \xrightarrow{\text{Delete}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \xrightarrow{\text{Replace}} \underset{\text{B}}{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}$$

Therefore, A *contains* B.

Containment and Avoidance for 0,1-Matrices

A B

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Delete Replace

Therefore, A *contains* B.

Definition

A *avoids* B: A does not *contain* B.

Extremal Function $0, 1$-matrices

Definition

ex(P, n): If P is a $0, 1$-matrix, then this is the maximum number of 1's we can have in an $n \times n$ matrix that $avoids$ P.

Example

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

ex$(P, 3) = 6$, because any 3×3 matrix with over 6 ones contains P.

Simple Result

Theorem

Let $P = \begin{pmatrix} 1 & 1 & 1 & . & . \\ 0 & 0 & 0 & & \\ 0 & 0 & 0 & & \\ . & & & . & \\ . & & & & . \end{pmatrix}$

where P has r rows and c columns.
Then, ex$(P, n) = n(r + c) - (r - 1)(c - 1)$ *Important: this is $O(n)$*

Containment and Avoidance for $0, 1...k$-Matrices

A

$$\begin{pmatrix} 2 & 0 & 1 \\ 3 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

B

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

Definition

A *contains* B: We can delete rows and columns in A, and replace numbers with smaller numbers to end up with B.

Containment and Avoidance for $0, 1...k$-Matrices

A

$$\begin{pmatrix} 2 & 0 & 1 \\ 3 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

B

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

Definition

A contains B: We can delete rows and columns in A, and replace numbers with smaller numbers to end up with B.

Definition

A avoids B: A does not *contain* B.

Extremal Function $0, 1...k$-matrices

Definition

$\text{ex}_k(P, n)$: The maximum sum of numbers an $n \times n$ $0, 1...k$-matrix A can have and still *avoid* P, where P is a $0, 1...k$-matrix.

Example

$$P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$\text{ex}_2(P, 3) = 14$, because any 3×3 matrix with all entries $0, 1, 2$ and sum over 14 contains P.

Mapping $0, 1...k$-matrices to $0, 1$-matrices

Definition

P_j: The $0, 1$-matrix formed by mapping all entries with values $\geq j$ to 1 and entries $\leq j - 1$ to 0.

Example

$$P \qquad\qquad P_2$$

$$\begin{pmatrix} 2 & 0 & 1 \\ 3 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$0, j$-matrices

Lemma

Let P be a matrix with only $0, j$ entries. For any $k \geq j$,
$\text{ex}_k(P, n) = (j-1)n^2 + (k-j+1)\text{ex}(P_j, n)$.

$0, j$-matrices

Lemma

Let P be a matrix with only $0, j$ entries. For any $k \geq j$, $\text{ex}_k(P, n) = (j-1)n^2 + (k-j+1)\text{ex}(P_j, n)$.

Reasoning.

The 'optimal' matrix that $avoids$ P has $j-1$ entries where an 'optimal' matrix that avoids P_j has 0's, and k entries where it has 1's. For example, an 'optimal' matrix that avoids some pattern P might look like:

$$\begin{pmatrix} j-1 & k & k \\ k & k & j-1 \\ k & j-1 & j-1 \end{pmatrix}$$

Calculating, this has $(j-1)n^2 + (k-j+1)\text{ex}(P_j, n)$ sum. □

Simple Inequality

Theorem

$(j-1)n^2 + (k-j+1)\text{ex}(P_j, n) \leq \text{ex}_k(P, n) \leq$
$(j-1)n^2 + (k-j+1)\text{ex}(P_1, n)$
where j is the maximum element in P.

Simple Inequality

Theorem

$(j-1)n^2 + (k-j+1)\text{ex}(P_j, n) \leq \text{ex}_k(P, n) \leq$
$(j-1)n^2 + (k-j+1)\text{ex}(P_1, n)$
where j is the maximum element in P.

Proof of LHS.

We find a matrix contained by P that has extremal function $(j-1)n^2 + (k-j+1)\text{ex}(P_j, n)$.

Let P' be the P with all non-j entries replaced with 0's.

$$\begin{pmatrix} j & 0 & 1 \\ j & j-1 & 1 \\ 0 & 0 & j \end{pmatrix} \rightarrow \begin{pmatrix} j & 0 & 0 \\ j & 0 & 0 \\ 0 & 0 & j \end{pmatrix}$$

□

Simple Inequality

Theorem

$(j-1)n^2 + (k-j+1)\text{ex}(P_j, n) \leq \text{ex}_k(P, n) \leq (j-1)n^2 + (k-j+1)\text{ex}(P_1, n)$

where j is the maximum element in P.

Proof of RHS.

We find a matrix that contains P that has extremal function $(j-1)n^2 + (k-j+1)\text{ex}(P_1, n)$.

Let P' be the P with all non-0 entries replaced with 1's.

$$\begin{pmatrix} j & 0 & 1 \\ j & j-1 & 1 \\ 0 & 0 & j \end{pmatrix} \rightarrow \begin{pmatrix} j & 0 & j \\ j & j & j \\ 0 & 0 & j \end{pmatrix}$$

□

Some 0,1,2-matrices

$$n^2 + \mathsf{ex}(P_2, n) \leq \mathsf{ex}_2(P, n) \leq n^2 + \mathsf{ex}(P_1, n)$$

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Example

$$\text{Let } P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$\mathsf{ex}_2(P, n) = n^2 + 2n - 1 \leftarrow$ analogous to lower bound

Some 0,1,2-matrices

$$n^2 + \mathrm{ex}(P_2, n) \leq \mathrm{ex}_2(P, n) \leq n^2 + \mathrm{ex}(P_1, n)$$

Example

Let $P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

$\mathrm{ex}_2(P, n) = n^2 + 2n - 1 \leftarrow$ analogous to lower bound

Example

Let $P = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$

$\mathrm{ex}_2(P, n) \geq n^2 + 3n - 2 \leftarrow$ NOT analogous to lower bound

We generally believe that the lower bound is closer than the upper bound though.

A general form

Theorem

$$\textit{Let } P = \begin{pmatrix} 2 & 2 & 2 & . & . & . \\ 1 & 1 & 1 & & & \\ 1 & 1 & 1 & & & \\ . & & & . & & \\ . & & & & . & \\ . & & & & & . \end{pmatrix}$$

$ex_2(P, n) = n^2 + O(n)$.

A general form

Theorem

$$\text{Let } P = \begin{pmatrix} 2 & 2 & 2 & . & . & . \\ 1 & 1 & 1 & & & \\ 1 & 1 & 1 & & & \\ . & & & . & & \\ . & & & & . & \\ . & & & & & . \end{pmatrix}$$

$ex_2(P, n) = n^2 + O(n)$.

Again serves as evidence that lower bound is better:

$$n^2 + \text{ex}(P_2, n) \leq \text{ex}_2(P, n) \leq n^2 + \text{ex}(P_1, n)$$
$$n^2 + O(n) \leq \text{ex}(P, n) \leq n^2 + O(n^{2-\epsilon})$$

Improving the Simple Inequality

Theorem

Let P be a $0,1,2$-matrix. The sum of numbers in a $n \times n$ matrix $avoiding$ P is at most $\leq n^2 + O(k\text{ex}(P_2, \frac{n}{\sqrt{k}}))$, where k is the number of 0's in the $n \times n$ matrix.
Can be modified to $n^2 + O(\sqrt{k}\text{ex}(P_2, n))$, easier to use but a bit weaker.

Improving the Simple Inequality

Proof Sketch.

Consider an $n \times n$ $0, 1, 2$-matrix that $avoids$ P. Build 'boxes' around the 0's as such:

$$\begin{pmatrix} 2 & 2 & 2 & 1 & 0 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 & 2 \\ 1 & 2 & 2 & 2 & 2 \end{pmatrix}$$

We can limit the number and side length of the boxes to get the desired result.

□

Implications

Theorem

Let P be a $0,1,2$-matrix. The sum of numbers in a $n \times n$ matrix $avoiding$ P is at most $\leq n^2 + O(k\text{ex}(P_2, \frac{n}{\sqrt{k}}))$, where k is the number of 0's in the $n \times n$ matrix.
Can be modified to $n^2 + O(\sqrt{k}\text{ex}(P_2, n))$, easier to use but a bit weaker.

- Bounds $\text{ex}_2(P, n)$ in terms of number of 0's in the $n \times n$ matrix that must $avoid$ P: $\text{ex}_2(P, n) \leq n^2 O(\sqrt{k}\text{ex}(P_2, n))$
- When k is (nontrvially) small, $O(\sqrt{k}\text{ex}(P_2, n))$ is better than $\text{ex}(P_1, n)$
- When k is large, it is not good enough.
- To make the result useful, we need to find another way to deal with the lots of k's case.

Further Research

Overarching Goal

Characterize all the extremal functions ex$_k(P, n)$ *in terms of 0,1 extremal functions* ex(P, n).

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- For 0,1,2-matrices, find ex$_2(P, n) - n^2$ upto a constant. We believe it is $\theta($ex$((P_2, n))$ rather than $\theta(P_1, n)$

Further Research

Overarching Goal

Characterize all the extremal functions ex$_k(P, n)$ in terms of 0,1 extremal functions ex(P, n).

- For 0,1,2-matrices, find ex$_2(P, n) - n^2$ upto a constant. We believe it is $\theta($ex$((P_2, n))$ rather than $\theta(P_1, n)$
- Find the exact value of ex$_2(P, n)$ where

$$\text{Let } P = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$$

Further Research

Overarching Goal

Characterize all the extremal functions ex$_k(P, n)$ in terms of 0,1 extremal functions ex(P, n).

- Generalize the theorem about a row of 2's followed by rows of 1's to $0, 1...k$-matrices, where we consider a row of i's followed by rows of j's where $i > j$.

Further Research

Overarching Goal

Characterize all the extremal functions ex$_k(P, n)$ in terms of 0,1 extremal functions ex(P, n).

- Generalize the theorem about a row of 2's followed by rows of 1's to $0, 1...k$-matrices, where we consider a row of i's followed by rows of j's where $i > j$.
- Generalize the last theorem (improvement on upper bound from simple inequality) to $0, 1...k$-matrices.

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